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NUMERICAL METHODS FOR INTERACTING BOUNDARY LAYERS

R.T. Davis* and M.J. Werle*

ABSTRACT

The problem of accurately calculating two dimensional high Reynolds number laminar separated flows is discussed for flows containing closed separation bubbles. Attention is directed both at the problem of identifying correct approximations to the Navier Stokes equations as well as at the problem of development of a numerical method for solving the resulting equations. The importance of asymptotic analysis to the resolution of these problems is discussed in two regards. First, asymptotic analysis provides the scale laws of the separation problem and thus helps set the finite difference mesh required. Second, asymptotic analysis helps in identifying a novel mechanism for upstream propagation of disturbances in interacting flows that numerical schemes must accommodate to be efficient. Solutions are presented for incompressible and supersonic separations obtained from interacting boundary-layer models which attempt to make use of the asymptotic results.

INTRODUCTION

It is well known that finding accurate solutions for the problem of high Reynolds number separated flow past a body is extremely difficult from both an analytical and numerical viewpoint. This is in contrast to the unseparated case which can now be handled routinely (at least with boundary-layer equations) since Prandtl [1] has provided the insight necessary for an asymptotic analysis which has been extended to higher order by Van Dyke [2] and others. Prandtl's analysis reveals the proper scaling of variables as Reynolds number goes to infinity and provides a set of boundary-layer equations which are relatively easy to solve numerically. The scaling of variables indicated by the boundary-layer analysis is also of benefit in the numerical solution of the full Navier Stokes equations.

If the flow separates, the problem becomes much more difficult. For many years those persons involved in numerical analysis have recognized difficulties with high Reynolds number separated flows but it is only recently that asymptotic analysis is beginning to reveal what the problems are and what to do about them. The difficulties appear in many forms. In boundary-layer theory the numerical solution may be limited by a square root singularity at separation as described by Goldstein [3]. Cell Reynolds number limitations and highly inaccurate, slowly converging solutions may limit the solution techniques for the full Navier Stokes equations. It is felt that the results of the asymptotic analysis for high Reynolds number will help overcome these type of difficulties.

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The asymptotic analysis for high Reynolds number separated flows is extremely difficult and is at this point still incomplete. However, there is some important information in the work which has been completed to date which is of use to those engaged in the numerical aspects of problems similar to the ones considered in the asymptotic analysis. We will not go into a review of the asymptotic results in this section but refer those who are interested to the excellent article by Stewartson [4] which is partially discussed in the next section.

Briefly, the asymptotic analysis reveals first that there are severe scaling problems around and downstream of a separation point. The mesh sizes required for numerical calculations must therefore be extremely small in those regions to achieve proper resolution. The asymptotic analysis reveals how small these mesh spacings must be from an order of magnitude viewpoint. Second, the asymptotic analysis reveals how the flow passes through a separation point (without a Goldstein singularity) and shows that the mechanism is provided by displacement thickness interaction. Finally, the asymptotic analysis reveals that the separation problem is boundary value in nature, even for supersonic flows, and gives some insight into how one should attack this boundary value problem numerically.

The purpose of this paper is to discuss how the person engaged in numerical analysis might make use of the results of the asymptotic theory. We have attempted to do this in several problems and have found that a combined analytical-numerical approach leads to a better understanding of the problem and a more efficient numerical method. At the same time we have been impressed by the extreme complexity of the high Reynolds number separated flow problem and feel that it will keep many of us busy for many years to come.

RESULTS FROM ASYMPTOTIC THEORIES

If one attempts to solve the entire flow field about a body using the full Navier Stokes equations, one has difficulty at high Reynolds numbers even for the unseparated flow case. This occurs partially because Prandtl's [1] boundary-layer theory dictates that two scalings are needed in the normal coordinate direction if a reliable and efficient calculation is to be made. While optimal coordinate theory or the removal of the outer inviscid portion of the flow field from the finite difference computation can partially relieve this difficulty (see Davis [5]), the more pertinent point is that the asymptotic analysis provides crucial information (the scale laws) for the calculation process.

Prandtl's theory provided the scaling appropriate for unseparated viscous flow past a body as Reynolds number goes to infinity. He showed that the boundary-layer region near the body surface scales like $Re^{-1/2}$ in the normal direction and order one in the flow direction. As a result of this scaling Prandtl provided a set of boundary-layer equations which are of the parabolic type. In this theory, the outer flow which drives the boundary layer is the undisturbed inviscid flow past the body and is governed by the Euler equations where variables are of order one and no scaling is necessary. The effect of the boundary-layer

displacement thickness on the outer flow is found to be a higher order effect which can be neglected in the first approximation. Prandtl therefore provided the numerical analyst with a simpler set of equations to solve. Once the inviscid flow past a body is provided, all quantities are available for solving the parabolic boundary-layer equations as an initial value problem starting at the stagnation point or leading edge and integrating downstream. Using a sufficient number of mesh points in a properly scaled set of boundary-layer equations, one can obtain highly accurate numerical solutions by using any one of a number of numerical methods, see Blottner [6]. The crucial point is that the calculations be performed in the scaled variables in order to obtain reliable solutions at high Reynolds numbers.

As is well known by now, Prandtl's theory (with a prescribed surface pressure distribution) encounters difficulty as a point of zero shear (separation) is approached and computations must terminate due to the appearance of the Goldstein [3] square root singularity. While it has not been proven that the Goldstein singularity must always appear, all numerical calculations which have been examined in detail reveal that as a point of zero shear on the body surface is approached, this singularity prevents the continuation of the solution into a reversed flow region (for further discussion of this point see for example, Brown and Stewartson [7], Werle and Senechal [8] and Werle and Davis [9]). Careful evaluation of higher-order boundary-layer solutions (see Van Dyke [2] and Werle and Davis [10]) shows a clear cut breakdown in the theory near the point where the first order boundary-layer theory predicts zero shear. The implication of this result is that the scaling postulated by Prandtl is not appropriate in this region and that new scaling laws must be sought to properly recover the solution to the Navier Stokes equations near separation.

The basis for establishing such a new set of scaling laws is not at all obvious but great inroads on this problem have followed directly from Lighthill's [11] analysis of the free interaction process that has long been observed to occur near separation points. Lighthill considered the possibility of the emergence of a sublayer at a point x_p^* in a flat plate laminar boundary layer in supersonic flow. He determined the scaling of the region and showed that an eigensolution is possible for such a sublayer and that through interaction with an external inviscid supersonic stream, it grows initially like e^{kx} where $k = 0.8272$ and $x = Re_{x_p^*}^{3/8}\,(x^*-x_p^*)/x_p^*$. Garvine [12] encountered similar branching in his attempt to solve the viscous shock-layer equations. Then Neiland [13], followed by Werle, Dwoyer and Hankey [14] found that eigensolutions exist for the hypersonic strong interaction region of a flat plate where algebraic growth ($x^{*\alpha}$ for $x^* \to 0$) instead of exponential growth is predicted. Brown, Stewartson & Williams [15] have identified the analytical link between these two solution families. The values of α were found to be very large ($\alpha \approx 50$ for adiabatic wall state and $\alpha \to \infty$ as $T_w/T_o \to 0$) clearly supporting the need for careful appreciation of the scale laws in this flow regime.

The first point of importance of the above results for the numerical analyst is that the eigensolutions grow with positive

exponents. Thus, in the supersonic or hypersonic flow problem these results give evidence that the interaction problem is boundary value in nature even though the flow field is governed by hyperbolic outer (inviscid) and parabolic inner (boundary layer) equations. In fact, if one attempts to numerically solve such interaction problems with forward marching, branches will be found to emerge as step sizes in the direction along the body surface are made small, see Werle et al [14]. Thus at high Reynolds number, there exists a mechanism for upstream propagation in a supersonic flow independent of the second derivatives in the streamwise direction appearing in the diffusion terms of the Navier Stokes equations.

The second important aspect of the new scale laws identified by Lighthill is that they provide the basis for a nonsingular description of separation. Stewartson and Williams [16] provided a consistant formulation of the supersonic separation problem using Lighthill's scaling in the lower (crucial) deck of a triple deck asymptotic description of viscous/inviscid interactions. With this formulation the dilemma of the Goldstein separation singularity is completely overcome and a smooth pass through the zero shear point into reverse flow regions occurs without incidence.

Stewartson [4] demonstrates the utility of the triple deck formulation of interactions by considering its application to the subsonic, transonic and hypersonic speed ranges for compressive or expansive interactions, injection studies, and trailing edge regions. Thus the triple deck approach appears to contain the key to understanding high Reynolds number separations in general. For example, Sychev [17], Messiter and Enlow [18] and Messiter [19] have shown that it is contained in a consistent model for steady incompressible high Reynolds number separation off a circular cylinder. While one might rightfully anticipate that downstream of separation the flow may become too complicated to allow the simplicity of a triple deck formulation, it seems reasonable to expect that this approach would be adequate over the entire separation bubble extent for the case of mild separations. This is the approach taken by Jenson, Burggraf and Rizzetta [20] and Burggraf [21] in their application of the triple deck concept to the supersonic ramp-induced separation problem. In that application the authors were able to capture the entire separation within the bottom deck and thus provide the first separation bubble solution for infinite Reynolds number.

For present purposes, the important issue is that the scaling for the separation problem is not the same as Prandtl's scaling - a fact that must be taken into account by the numerical analyst. The results of Stewartson's [4] analysis indicates that in order to obtain proper resolution near a separation point, one must take mesh sizes of the order of magnitude of $Re_{x_p^*}^{-3/8}$ in the streamwise direction and of the order of magnitude of $Re_{x_p^*}^{-5/8}$ in the normal direction, see Fig. 1. Since the boundary-layer equations are usually scaled in the normal direction by $Re_{x_p^*}^{-1/2}$ according to Prandtl's theory, one needs to use step sizes in the normal direction near the wall of the order $Re_{x_p^*}^{-1/8}$ in the boundary-layer

equations near separation to achieve proper resolution while the streamwise steps must be of order $Re_{x_p^*}^{-3/8}$. Thus the most severe resolution problem at high Reynolds number comes in the streamwise direction when using boundary-layer equations. A variable mesh system in the boundary-layer equations can be used to overcome this difficulty.

In addition to providing the proper scaling near separation, Stewartson's triple deck analysis contains an explanation for how upstream propagation occurs through interaction and gives a key to the numerical analyst on how to deal with it. Stewartson's first order bottom deck equation for flow past a flat plate in terms of variables of order one near a point x_p^* on the plate surface are given as follows (see Stewartson [4], p. 168). The variables u, v, x, y and p are the usual ones used in boundary-layer analysis, except that they have been scaled appropriately for the bottom deck.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} , \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{2}$$

$$u, v = 0 \quad \text{at} \quad y = 0 , \tag{3}$$

$$u \to y \quad \text{as} \quad x \to -\infty , \tag{4}$$

$$u \to y - \delta(x) \quad \text{as} \quad y \to \infty , \tag{5}$$

where

$$p = \frac{d\delta}{dx} \tag{6}$$

for the supersonic case, and

$$p = - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\delta'(x_1)}{x - x_1} \, dx_1 \tag{7}$$

for the subsonic case. The quantity x is a properly scaled variable measured from the point x_p^* on the plate surface and $\delta(x)$ is a displacement thickness. It should be noted that the above are boundary-layer equations with boundary conditions which are different from the usual ones.

We can see the boundary value nature of the problem in the streamwise direction by evaluating Eqs. 1-7 at the lower deck edge. At the edge

$$u = y - \delta(x) \tag{8}$$

and thus from the continuity Eq. 2

$$\frac{\partial v}{\partial y} = \frac{d\delta}{dx} \tag{9}$$

which integrates to

$$v = \frac{d\delta}{dx}\,[y - C(x)] \quad . \tag{10}$$

Evaluating the momentum Eq. 1 at the edge results in

$$\frac{dp}{dx} - [C(x) - \delta]\frac{d\delta}{dx} = 0 \quad . \tag{11}$$

For the supersonic case $p = \frac{d\delta}{dx}$ and Eq. 11 results in

$$\frac{d^2\delta}{dx^2} - [C(x) - \delta]\frac{d\delta}{dx} = 0 \tag{12}$$

This last relation (Eq. 12) indicates that the supersonic problem becomes boundary value entirely through the displacement thickness interaction. The quantity C is a function of x but its initial value is positive (C = 0.8272 from Lighthill's linearized analysis) indicating a general form of the solution of Eq. 12 with a growing exponential character. This indicates that a downstream boundary condition is required for the problem to be well posed.

The subsonic interaction problem is more complicated since the dp/dx term in Eq. 11 is related to a Cauchy integral through Eq. 7. However, in that case there is no doubt nor surprise in finding that the problem is boundary value. More importantly though, it is now clearer what the actual mechanism is for this upstream propagation - a result that must be accommodated in numerical studies of subsonic interactions.

Not all of the insight gained from the asymptotic analysis of interactions has yet fully been implemented in the general study of separated flows. It is believed to be extremely significant that an examination of these equations reveals that there are no terms appearing which are not included in the ordinary boundary-layer equations adjusted for displacement interaction. This provides support to those who have been employing the interacting boundary-layer concept for many years plus provides new information concerning future directions of study. The formal link between the two approaches has been clearly identified by Burggraf, Werle, Rizzetta and Vatsa [22] (see also Vatsa [23] and Rizzetta [24]) who showed that, while the interacting boundary-layer model does apply well at finite Reynolds numbers, its scaling is accurately predicted by the triple deck formulation. This is shown in Fig. 2 where the surface shear stress distribution (normalized with its flat plate value) obtained from the interacting boundary-layer equations is compared with the triple deck solution. This comparison was performed for a compression ramp configuration with the reduced ramp angle α, as defined in the figure, held fixed at 2.5 as $Re \to \infty$. The important issue is that the asymptotic solution reasonably predicts the scale of the interacting boundary-layer solutions even at $Re = 10^4$. It would be helpful then to employ the asymptotic solution to scale the finite difference mesh for all subsequent ramp type separations. This is the approach taken by Werle [25] in a detailed study of scaling for supersonic/hypersonic ramp separations.

Tu and Weinbaum [26] have since shown that the principle difference between these two models occurs in the middle deck region which basically acts as a rotational inviscid flow. They

verify that allowing for the effect of streamtube divergence in this region extends the range of applicability of the multi-deck approach to much lower Reynolds number ranges than originally indicated. Thus, it is now believed that the small separation bubble problem can be reliably addressed with the interacting boundary-layer concept so long as one accounts for the scaling levels predicted by the triple deck formulation

Our present solution techniques have not yet fully implemented the conclusions drawn from the lower deck equations. We have attempted to recognize that the interacting boundary-layer problem is boundary value in nature and have oriented the numerical schemes accordingly. For example, if one traces back through the equations to determine the origin of the $d\delta/dx$ term in Eq. 12, it is found that it comes from the $\partial u/\partial x$ term in momentum (Eq. 1) and the $\partial u/\partial x$ term in continuity (Eq. 2). One would then expect that a forward difference of $\partial u/\partial x$ term in the continuity equation (the major contributor) would be advantageous in regions of a flow where interaction is important. We have found this to be the case in some calculations and have found that this may greatly enhance numerical convergence rates. To do the problem properly one needs to scale the major part of the $\delta(x)$ dependence out of the problem and forward difference the $\delta'(x)$ coefficients only in regions of the flow where this should be done.

In addition, we have tried to account for the boundary value nature of the problem by adopting difference schemes originally suited to the solution of elliptic partial differential equations. In particular, Werle and Vatsa [27] took an approach that in effect adds a fictitious time derivative to Eq. 12 and then solves it as a time dependent problem with an ADI (alternating direction implicit) scheme, i.e. replace Eq. 12 with

$$\frac{\partial^2\delta}{\partial x^2} - [C(x,t) - \delta]\frac{\partial\delta}{\partial x} = \frac{\partial\delta}{\partial t} \quad . \tag{13}$$

In terms of the boundary-layer equations, this is equivalent to replacing the pressure gradient term, dp/dx with $\partial p/\partial x - \partial\delta/\partial t$ and developing an ADI scheme on δ alone, with the $\partial^2\delta/\partial x^2$ terms being written as central differences. If $C - \delta$ is positive in Eq. 13 one would write the term $\partial\delta/\partial x$ as a forward difference to insure diagonal dominance in the inversion of the algebraic equations which result. A somewhat similar time dependent approach was also used by Jenson, Burggraf and Rizzetta [20] but there the entire flow was set in a time dependent frame.

Finally, at the same time that we have been doing interacting boundary-layer calculations, we have also been doing implicit Navier Stokes calculations. It has consistently been found that a better understanding of the boundary-layer solution techniques leads to a more efficient Navier Stokes calculation at high Reynolds number. Since the boundary-layer equations are the limiting form of the Navier Stokes equations, this statement should be obvious but it seems to be often overlooked. For example, Stewartson's [4] analysis indicates that a mechanism of upstream propagation exists in the Navier Stokes equations which in many problems is much more important than the upstream propagation due to the second derivatives in the streamwise direction. It is

important that we more fully understand this mechanism in order to produce the most efficient algorithms. In addition, the asymptotic analysis provides the scalings necessary to determine mesh sizes necessary to achieve proper resolution of the flow field in Navier Stokes as well as interacting boundary-layer calculations.

SUPERSONIC INTERACTING BOUNDARY LAYERS

The basic problem addressed in this section is the solution of the interacting boundary-layer equations for supersonic flow. These equations are given by the classical boundary-layer equations except that the local inviscid properties are taken to be those occurring on the displacement surface produced by the viscous flow. Additional higher-order boundary-layer effects such as curvature and vorticity interaction corrections are not being considered at this juncture. This approach is not at all new and was pioneered by Lees and his colleagues [28] as well as by Baum [29], [30], Weinbaum and Garvine [31], Holden [32] and many others.

To proceed it is important to first recall from the previous section that these equations contain all the elements of the triple deck formulation for the small separation bubble problem thus allowing a direct carry over of the scale laws and nature of the problem. In particular the solution technique applied here involves an attempt to better appreciate the boundary value nature of the interaction problem. Many previous analytical efforts, starting with Lighthill [11], have clearly shown that the interacting equations communicate information forward by rather novel (and not yet fully understood) mechanisms. One such mechanism stems from the pressure gradient term itself, which as shown in Eq. 12, produces a second order differential equation in δ. More importantly, this equation has exponentially growing eigensolutions in the positive x direction, which would be expected to cause numerical difficulties unless one downstream boundary condition were imposed. Thus, the point of concern in the present formulation is the proper handling of this pressure gradient term.

For laminar flows it is reasonable to first rewrite the governing equations in Levy-Lees type variables ξ and η for the longitudinal and normal coordinate directions. The velocity and temperatures are normalized with their edge values as $F = u/U_e$ and $\theta = T/T_e$ so that with the added (nonrestrictive) assumption of a linear viscosity law, the longitudinal momentum equation becomes

$$F_{\eta\eta} - VF_{\eta} + \beta(\theta - F^2) - 2\xi FF_{\xi} = 0 \tag{14}$$

with V given by the continuity equation and θ by the energy equation. The pressure gradient term β is given as $\beta=(2\xi/U_e)(dU_e/d\xi)$ which is coupled to the inviscid flow field through influence of the displacement thickness growth on the inviscid stream. This is accounted for by the local deflection of the inviscid stream, α_T, given by the relation

$$\alpha_T = \alpha_s + \tan^{-1}(d\delta^*/ds) \tag{15}$$

where α_s is the local surface inclination and s is the distance along the surface. The relationship between U_e and α_T provides the final statement that

$$\beta = K(\xi)\ \frac{dU_e}{d\alpha_T}\ [\frac{d\alpha_s}{ds} + \frac{1}{1+(\frac{d\delta^*}{ds})^2}\ \frac{d^2\delta^*}{ds^2}] \tag{16}$$

where the term $dU_e/d\alpha_T$ is obtained from the inviscid flow law (i.e., linear theory, tangent wedge law, etc.).

The last term in this relation, $d^2\delta^*/ds^2$, is the present counterpart of the term $d^2\delta/dx^2$ in Eq. 12 and provides the path for information flow from downstream regions. This boundary value nature of the problem is accommodated by carrying over the standard time-like solution technique used in solving elliptic partial differential equations (the details of the finite difference scheme used in the present application have been presented by Werle and Vatsa [27]). The principle issue is the time-like formulation of the pressure gradient term through the redefinition that $\bar{\beta} = \beta + a(\partial\delta^*/\partial t)$ where the function $a(\xi)$ was introduced for numerical convenience and should have no bearing on the ultimate steady state solution achieved. With this time-like term, the solution can be marched forward in the t-plane with an ADI approach until $\partial\delta^*/\partial t \to 0$, the important point being that this allows one to hold a downstream boundary condition on δ^* for all time. For current applications either the pressure or pressure gradient are set at their expected downstream inviscid levels - this being implemented as a derivative boundary condition on δ^* through the pressure interaction equation. This simple modification when made with care should allow one to do mildly separated flows using any good implicit boundary-layer solution technique. As was pointed out in the previous section, it appears that additional refinements may improve the technique. For example, there is evidence that forward differencing of some terms could be advantageous. This point must be looked at carefully in future studies of this problem.

We note in passing that the approach outlined above is not restricted to the finite difference formulation of the problem but rather to the boundary value nature of the problem itself. Thus, any solution technique applied to the interacting equations should benefit from this concept. In particular, one would anticipate that this approach would be useful for the approximate integral solution methods and should provide a means of eliminating the artifice of a supercritical/subcritical jump presently being used in those schemes. Also, the use of this approach for solution of the viscous shock layer equations (a form of interacting boundary-layer equations) should be useful (see Werle, Srivastava, and Davis [33] for further discussion of this point).

There are numerous examples of the successful solution of the interacting boundary-layer equations in the open literature and thus this is not the point of concern here. In the two example applications that follow, it is first shown that the resulting solutions to this approximate set of equations essentially reproduce availabe solutions to the full Navier Stokes equations and agree well with reliable experimental data in the laminar supersonic flow regime. In addition, it is shown that this approach allows one to consider extremely complicated flow geometries that would otherwise be difficult to handle.

The first example involves flow past a 10° compression ramp placed in a $M_\infty = 4$ mainstream with $Re_\infty = 6.8 \times 10^4$ and an adiabatic wall temperature. For this case Lewis et al [34] have provided experimental results and Carter [35] has obtained accurate numerical solutions to the Navier Stokes equations - these latter results providing the bench mark against which the interacting boundary-layer model must be compared. Comparison of the present calculations (see Vatsa [23] for details of the numerical solutions) are presented in Fig. 3. These results were obtained with an imposed pressure gradient of zero at the downstream boundary ($s = 2.0$) and were computed to the same level of numerical accuracy as that of Carter's (i.e. the truncation errors were of $O(\Delta\eta^2)$ and $O(\Delta s^2)$ in both schemes). In addition, Vatsa's results were obtained with an approximate representation of the inviscid flow (a generalized supersonic/hypersonic tangent wedge law) and a rapid but smooth (versus discontinuous) change in surface slope at the ramp hinge line. The principle point of interest is that the solutions obtained from the interacting boundary-layer equations essentially reproduce those of the Navier Stokes equations. The fact that both solutions over-predict the experimental results is believed to be due to the presence of an axisymmetric relief effect in the experimental set up (see Horton [36] or Vatsa [23] for further discussion). Also it is worthy of note that two major difficulties which often plague other solution methods have not been encountered here. First, instead of a supercritical/subcritical jump, the present results produce a smooth self-imposed transition of the upstream weak interaction (supercritical) boundary layer over to a strong interaction (subcritical) region with subsequent return to a weak interaction region downstream. Second, the problem of branching attendant to all applications of initial value techniques to this problem (see Reyhner and Flügge-Lotz [37]) was avoided.

Numerous other applications of this approach have now been conducted and are available for review in the literature (see [25], [27], [38], and [39]). Such solutions are not meant to be definitive, but rather indicative of the generality of the numerical concept, a generality, best exemplified by the recent results achieved by Polak et al [39] in the study of supersonic separated flow over protuberances.

A relatively simple yet practically meaningful configuration studied was that of supersonic flow over a train of sinusoidal protuberances resulting in the occurrence of multiple separation bubbles. Although solutions have been presented by others for attached flow cases (see Fannelop and Flügge-Lotz [40], Benjamin [41], and Inger and Williams [42]) no separated flow results have been previously given.

The separated flow results of Polak et al [39] are shown in the Figs. 4 and 5. For these calculations the downstream pressure gradient was held fixed at zero during every time step of the relaxation process. As shown in Fig. 4, this results in the appearance of up to as many as three separation bubbles for the 4 wave case. Figure 5 shows the effect of interaction on the pressure signature and boundary-layer displacement thickness for this case. The pressure deviates from the inviscid sinusoidal distribution after separation takes place (after the first hump)

in response to the fact that the displacement thickness effectively fills the depressions in the plate. Downstream the flow appears to be heading toward a new flat plate solution that could not have been achieved without using the interaction method to pass through the separation region. Considering the overall complexity of this flow it is difficult to see how one would have achieved this solution with any method that did not directly use a downstream boundary condition.

SUBSONIC INTERACTING BOUNDARY LAYERS

For this flow regime, the entire approach to the problem has historically been quite different from that used in the supersonic problem. This, most certainly, comes about due to the more "global" effect any subsonic interaction produces on the inviscid flow. However, with the new insight gained from the triple deck formulation of the problem, it is clear that the two flow regimes hold more in common than one might have anticipated. Thus, an ADI approach similar to that discussed in the previous section, but which uses the Cauchy integral instead of the linear supersonic interaction law to evaluate dp/dx should be explored.

Work on the subsonic interaction problem has been initiated by Carter [43], Carter and Wornom [44], Klineberg and Steger [45] and Briley and McDonald [46]. Most methods though have shown slow numerical convergence rates indicating room for significant improvements. It is believed that the bases for such improvements should come from better understanding of the basic mechanisms for interacting flows such as is available from asymptotic theories.

For several years, we have been doing Navier Stokes type calculations for simple geometry flows using an ADI technique on the stream function and vorticity equations in conformal coordinates. This technique is especially useful for high Reynolds number flows. It uses similarity type variables and is designed such that the boundary-layer like terms are totally contained in one sweep of an ADI scheme (see Davis [47], U. Ghia and Davis [48], and Davis [5] for example) with the second sweep essentially performing displacement adjustments to the inviscid flow. We have found that in several separated flow cases considered (see U. Ghia and Davis [48] and Werle and Bernstein [49]) the nonparabolic terms in the vorticity equation have a negligible influence on the solution, even at fairly low Reynolds number. The resulting model, which uses the full stream function (elliptic) equation and an approximate vorticity (parabolic) equation certainly represents one member in a large family of interacting boundary-layer models. Since further simplifications to this model are possible, we have called this simply a parabolized vorticity model of the full Navier Stokes equations. From an asymptotic viewpoint, the parabolized vorticity model is valid to second order in the entire flow field for attached flows or flows with small separation bubbles. Our feeling is that the effectiveness of the parabolized vorticity model is directly related to the coordinate system chosen in which to make the approximation and thus, care must be exercised in this regard. The proper coordinates are perhaps related to the type of optimal coordinates identified by Davis [5], but more work is needed in this area.

While the use of this parabolized vorticity model has enjoyed some success to date, its method of handling the stream function equation is more involved than felt necessary. In particular a true interacting boundary-layer model should also be able to employ a simplified version of the stream function equation in the viscous region along with coupling with the inviscid stream through displacement thickness. For the incompressible case the inviscid flow is most easily handled with an integral method which uses source-sink or vortex distributions on the body surface. This is the basic approach that will be discussed here and that was also followed by Carter [43], Carter and Wornon [44] and Briley and McDonald [46].

In order to study incompressible interacting boundary layers, we have concentrated on two dimensional flows past semi-infinite bodies which are easily transformed to a stagnation point type flow by a conformal transformation (see U. Ghia and Davis [48] and Davis [5]). If we let $z = x+iy$ be the physical plane and $\zeta = \xi+i\eta$ be the conformal plane, see Fig. 6, the conformal transformation is given by $z = f(\zeta)$. The scale factors appearing in the coordinate transformations become $h_1 = h_2 = h = |f'(\zeta)|$. We can easily write down the Navier Stokes equations in conformal coordinates, see Lagerstrom [50] p. 66 for example. Next we introduce similarity type variables by defining the stream function as $\psi = \xi f(\xi,\eta)$ and the vorticity as $\omega = -\xi H g(\xi,\eta)$ where $H = 1/h^2$. Substituting these expressions into the full Navier Stokes equations and rewriting them in boundary-layer variables, one finds that in the viscous region the terms which would make the equations elliptic in space are of third order. Neglecting those elliptic third order terms results in the parabolized vorticity equation and a stream function equation as follows:

$$g_{\eta\eta} + [2\frac{H_\eta}{H} + f + \xi f_\xi]g_\eta + [\frac{2}{\xi}\frac{H_\xi}{H} + \frac{H_\eta^2+H_\xi^2}{H^2} + \frac{H_\eta}{H}(f + \xi f_\xi)$$

$$- f_\eta(1 + \xi\frac{H_\xi}{H})]g + [2\frac{H_\xi}{H} - \xi f_\eta]g_\xi = \frac{1}{H}\frac{\partial g}{\partial t} \qquad (17)$$

and

$$f_{\eta\eta} = g \quad . \qquad (18)$$

The definition of the dimensionless variables is given in U. Ghia and Davis [48]. These equations are formally valid to second-order (see Van Dyke [2]) even though they contain some third order terms in the coefficients of the vorticity Eq. 17. These terms are not neglected since no advantage is gained from a numerical viewpoint. An examination of Stewartson's [4] triple deck formulation of separation will show that it contains no new terms and one would expect that Eqs. 17 and 18 hold for mild separations if they are allowed to interact with an external inviscid stream.

The appropriate boundary and matching conditions are

$$f(\xi,0) = 0 \quad , \qquad (19)$$

$$f_\eta(\xi,0) = 0 \quad , \tag{20}$$

$$g(\xi,\eta) \sim 0 \quad \text{as} \quad \eta \to \infty \quad , \tag{21}$$

and one of the following or some other condition. Prescribe either

$$f_\eta,\ f,\ \eta f_\eta - f, \ \text{or ?} \quad \text{as} \quad \eta \to \infty \quad . \tag{22}$$

As shown the outer edge matching condition (Eq. 22) can be handled in several ways. For example, if we first let $f = \eta + h$ to remove the first order inviscid flow then we can show from asymptotic matching that

$$\left.\begin{aligned} \eta h_\eta - h = D(\xi) &= \int_0^\infty \eta g d\eta \quad , \qquad (23) \\ h_\eta &= h_{i\eta}(\xi,0) \quad , \qquad (24) \end{aligned}\right\} \text{as} \quad \eta \to \infty$$

and

$$h_i\ (\xi,0) = -D(\xi) \tag{25}$$

where $h_{i\eta}(\xi,0)$ and $h_i(\xi,0)$ are the inviscid values evaluated on the body surface and $D(\xi)$ is a displacement thickness like function.

Thus, if we use a surface source distribution to solve the inviscid flow field we need only solve the Cauchy integral equation evaluated at $\eta = 0$ in the transformed plane,

$$\xi h_{i\eta}(\xi,0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - x} \frac{d}{dx} (xD(x))\ dx \quad , \tag{26}$$

to produce, for example, an edge condition for h of the form

$$h = \eta h_{i\eta}(\xi,0) - D(\xi) \quad \text{as} \quad \eta \to \infty \quad . \tag{27}$$

The displacement quantity $D(\xi)$ is given by Eq. 23.

If instead one chooses to use a surface vortex distribution, the inverse Cauchy integral equation evaluated at $\eta = 0$ in the transformed plane gives

$$[\xi D(\xi)]_\xi = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - x}\, x\, h_{i\eta}(x,0)\ dx \quad , \tag{28}$$

which produces for example an edge condition for $\eta h_\eta - h$ of the form

$$\eta h_\eta - h = D(\xi) \quad \text{as} \quad \eta \to \infty \quad . \tag{29}$$

The Eqs. 17 and 18 with boundary and matching conditions (Eqs. 19-22) and either Eqs. 26 and 27 or Eqs. 28 and 29 form a closed set.

The numerical method of solution used has been as follows. Equations 17 and 18 are solved in a time dependent sense. Equation 17 has been quasilinearized about the previous time step (see Blottner [6] pp. 3-18) which results in Eqs. 17 and 18 being a coupled set of linear equations at a given time level. All derivatives are written in the normal finite difference sense for implicit boundary-layer calculations marching in the ξ direction with upwind differencing on the g_ξ term. Initial conditions for the flow field are prescribed and Eqs. 17 and 18 are integrated at the next time step by solving the difference equations starting from the stagnation point and proceeding to downstream infinity with the boundary conditions (Eqs. 19-21) prescribed along with f as $\eta \to \infty$. This results in a displacement thickness distribution $D(\xi)$. This distribution is substituted into Eq. 26 to calculate a $h_{i_\eta}(\xi,0)$ using the method described by Hess [51]. Equation 27 is used to evaluate $h(\xi,\eta)$ as $\eta \to \infty$ which serves as an outer boundary condition for $f = \eta + h$ as $\eta \to \infty$ for the next cycle of the time dependent process. The process is repeated until a steady state type solution is achieved.

The calculations were done in the independent variables S,N which transform the infinite region to a unit square, see U. Ghia and Davis [48]. As an example of the application of the method, we use the flow past the bodies shown in Fig. 7. This body shape was also considered by U. Ghia and Davis [48] for their Navier Stokes calculations, which provide a basis of comparison for the present study. The parameter α determines the body shape. As α goes to zero the body becomes a flat plate of finite thickness and as α goes to one the body becomes a parabola. Thus, through the parameter α, one can consider flows with fairly strong, mild, and weak separations. We choose values of α of 0.1 and 0.05 for the present discussion. The figure also shows values of the quantity S, the transformed coordinate.

The calculations for surface skin friction shown in Fig. 8 are for $\alpha = 0.1$. In the calculations, 40 steps were taken in the S direction and 100 steps were taken in the N direction. Without interaction the adverse pressure gradient downstream of the corner is sufficient for the solutions to terminate with a Goldstein singularity. The calculations shown for the interacting boundary-layer are compared with Navier Stokes solutions using the same step sizes, difference schemes etc., wherever possible. One sees that the two calculations are in excellent agreement down to a Reynolds number based on nose radius of curvature of about 1000.

If one decreases the step sizes in the S direction, one will find that these calculations are not accurate at high Reynolds number because of insufficient grid points near the corner (see Davis [5]). However, they are presented here to show that when equivalent calculations are made, interacting boundary-layer solutions tend to agree with the Navier Stokes calculations. To do the problem properly, we need to honor the $Re^{-3/8}$ scaling indicated by the asymptotic analysis for the region around the corner where separation may occur. If this is done the flow will probably show a separated region at high Reynolds number.

In the present calculations it was found that the most

sensitive parts of the above process have been related to how one prescribes the outer edge condition (Eq. 22) and how one differences the f_ξ terms in the vorticity Eq. 17.

Carter [43] has found that his technique works best by prescribing $\eta h_\eta - h$ at the edge which relates to the displacement thickness through an equation like Eq. 29. He then effectively finds $h_{i\eta}(\xi,0)$ from the boundary-layer calculation and uses an equation like Eq. 28 to find $D(\xi)$ for the next time step in the calculation. We have found that while this works the convergence rate of our own method is made faster by prescribing h (streamfunction) at the edge. It would appear that the quantity one does not want to prescribe is h_η at the edge since this is equivalent to prescribing outer edge velocity which can lead to the emergence of a Goldstein type singularity. We have also not experienced the problem with diagonal dominance reported by Carter [43] and one is able to run calculations with almost any time step, even though a particular value of time step will lead to an optimal convergence rate.

We have found that for strongly interacting flows, convergence can be greatly enhanced by writing the f_ξ terms in the vorticity Eq. 17 as forward differences. This can be defended on the basis of Eq. 12. We have not yet fully decided when this should or should not be done, but we are sure that the basis for doing it has to do with the importance of propagation of information upstream through displacement thickness interaction. The same changes made in our full Navier Stokes calculations, produced similar improvements in numerical convergence rates.

In Fig. 9 interacting boundary-layer solutions for surface skin friction are shown for $\alpha = 0.05$, using the same mesh sizes as in the previous case. This case shows separation as Reynolds number is increased. Admittedly the resolution is poor at high Reynolds number due to the fact we have not yet built the triple deck scaling into these calculation schemes. We experience difficulties in convergence at Reynolds numbers higher than the ones shown. Whether this is due to poor resolution, a deficiency of the numerical method, or a signal of the breakdown of interacting boundary-layer theory, is not known at the present time.

The reason that we have not progressed further with the subsonic problem is that we have felt that it is time for several refinements in our present method for computing incompressible interacting flows. We are presently attempting to incorporate the ADI scheme on displacement thickness into the method and are also incorporating a variable mesh system to gain better resolution near separation as triple deck analysis would indicate we need.

CONCLUSIONS

The first conclusion of this paper is that the interacting boundary-layer equations, when written in an appropriate coordinate system, are adequate for predicting high Reynolds number flows involving small separation bubbles. The second conclusion is that in order to develop efficient and accurate numerical schemes for high Reynolds number separated flows, the results of asymptotic analysis must be accommodated.

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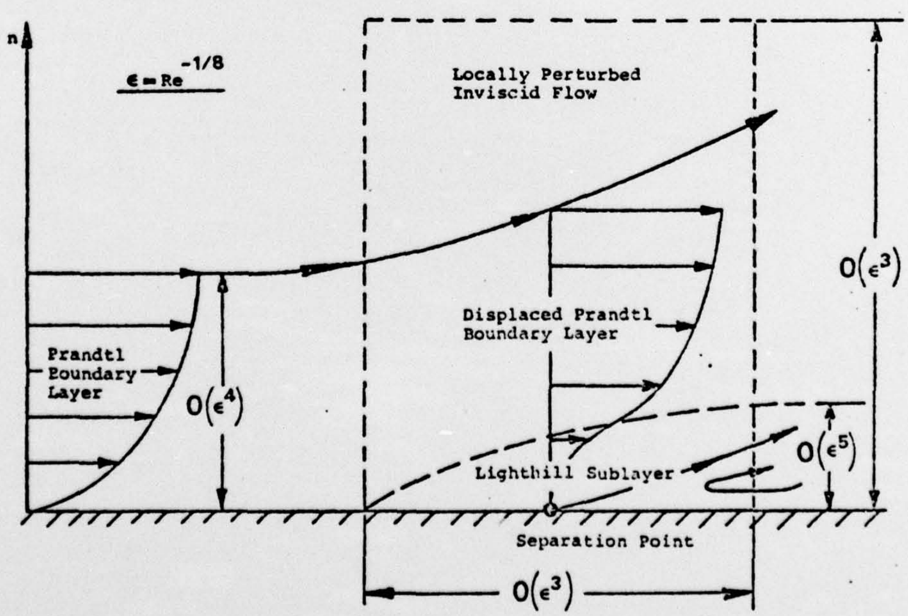

Fig. 1. Scaling near laminar separation at high Reynolds number, after Stewartson [4].

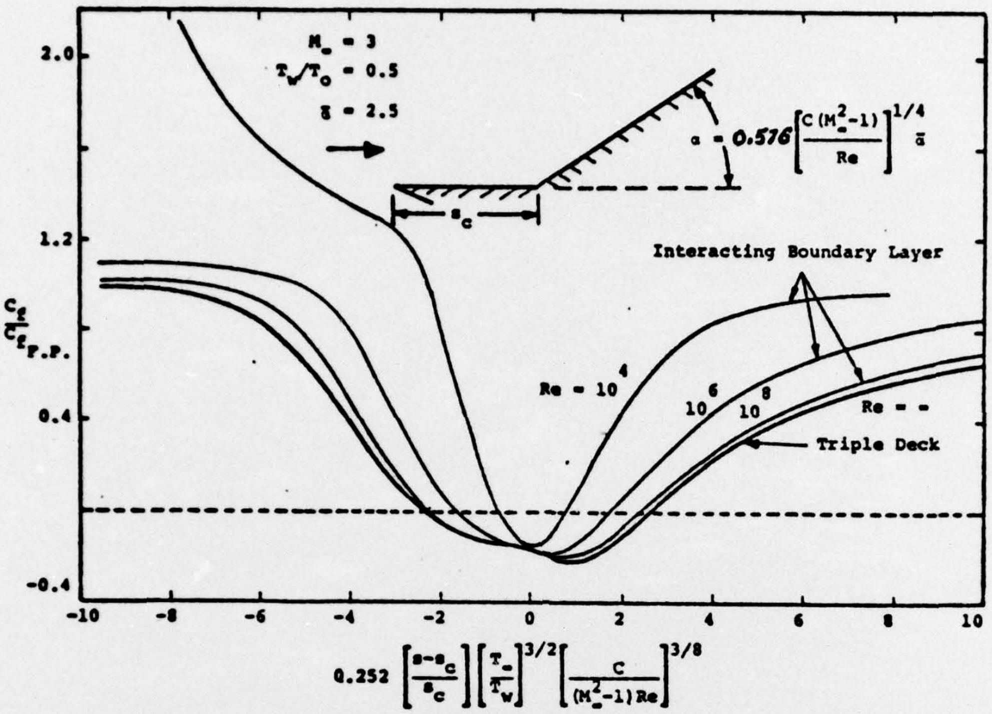

Fig. 2. Comparison of asymptotic and interacting boundary-layer solutions, after Burggraf et al [22].

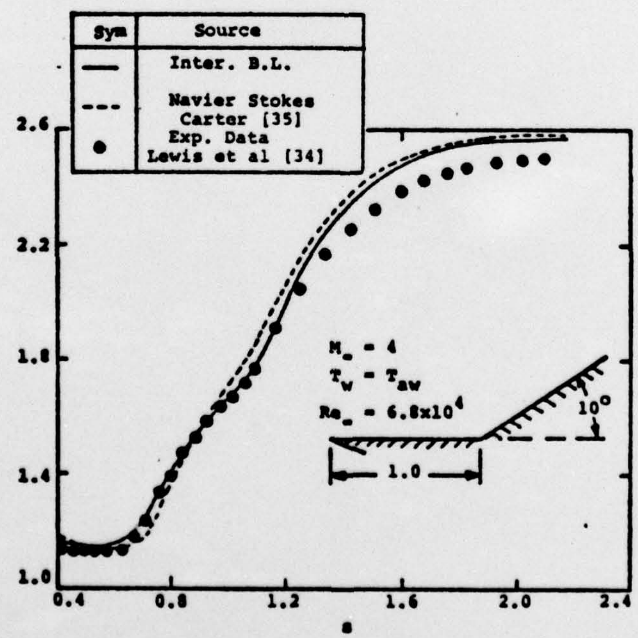

Fig. 3. Comparison of supersonic interacting boundary-layer with Navier Stokes solutions for a compression ramp, after Vatsa [23].

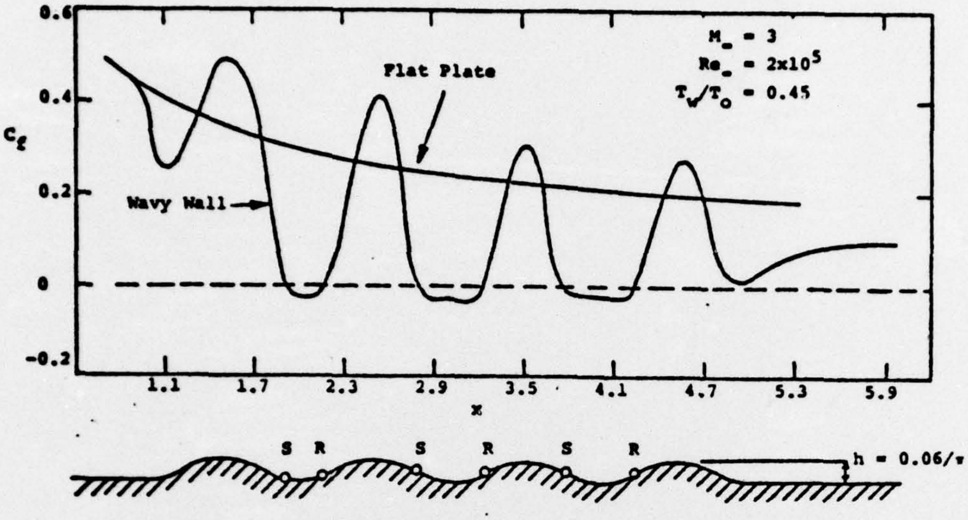

Fig. 4. Effect of multiple waves on supersonic separation, after Polak et al [39].

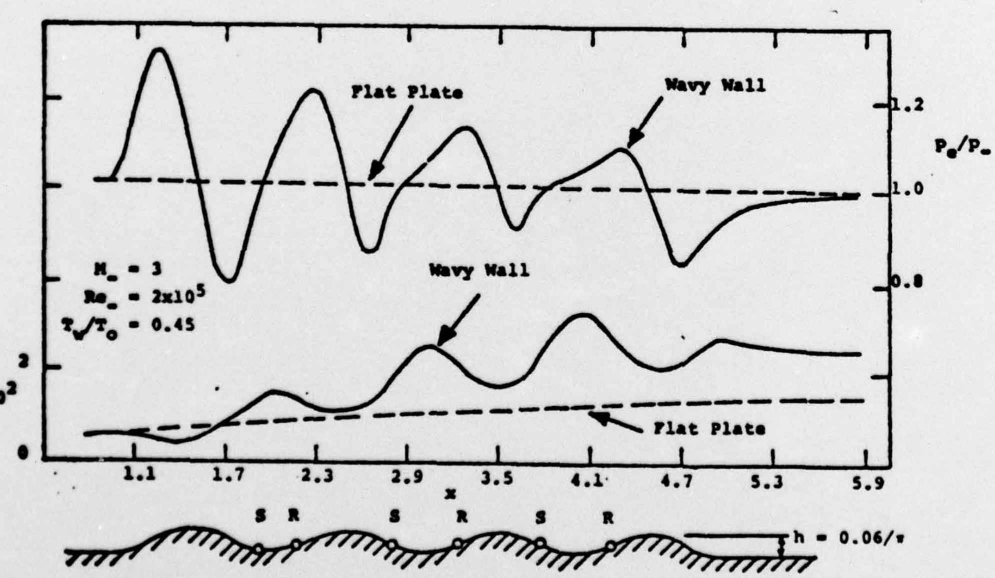

Fig. 5. Distribution of pressure and displacement thickness along the wavy wall, after Polak et al [39].

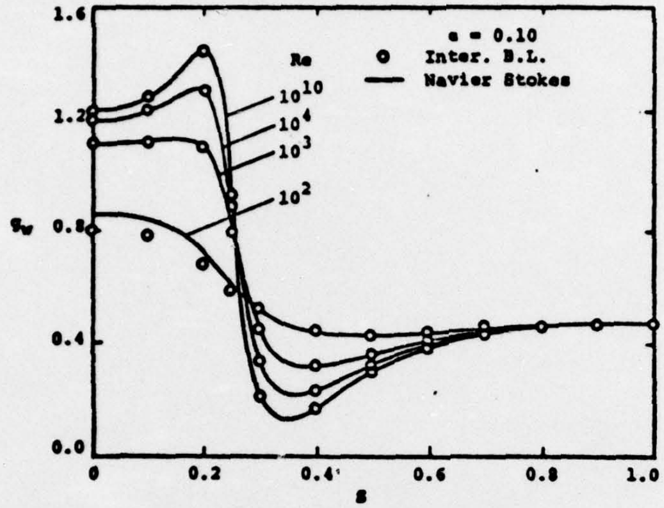

Fig. 8. Comparison of incompressible interacting boundary-layer with Navier Stokes solutions for blunt plates, after Davis [5].

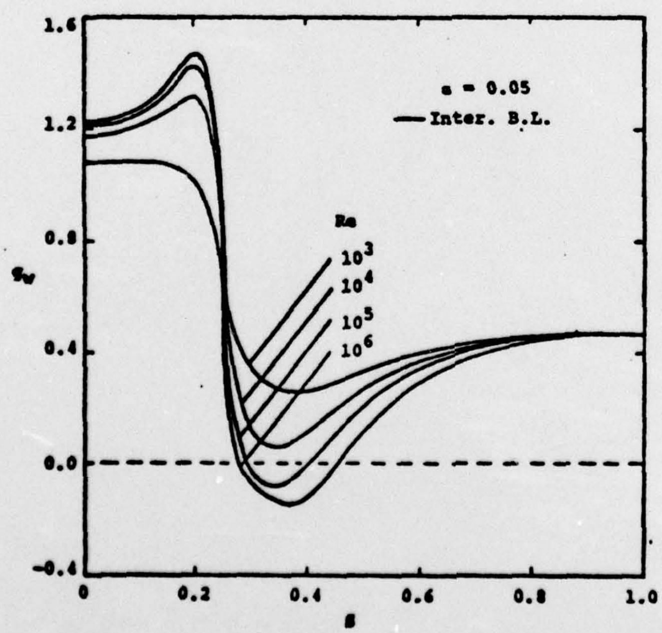

Fig. 9. Incompressible separating boundary-layer solutions for blunt plates.

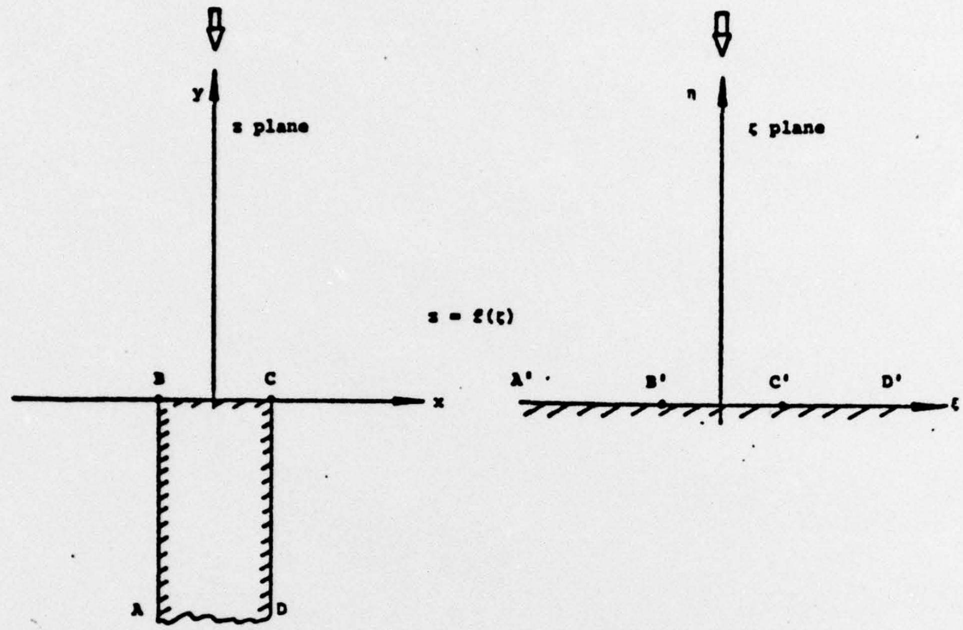

Fig. 6. Transformation from physical to conformal plane, after U. Ghia and Davis [48].

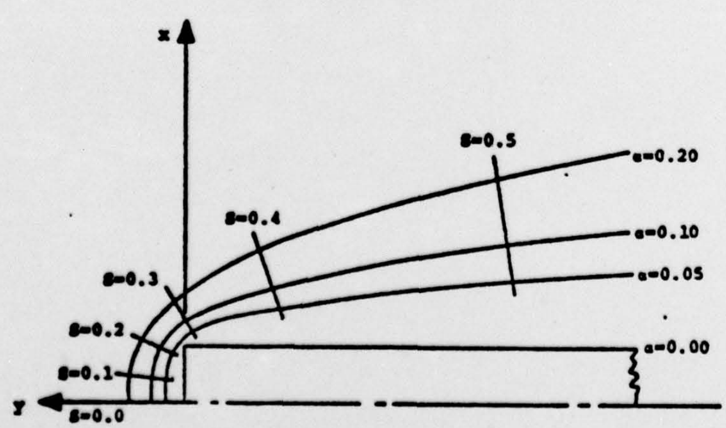

Fig. 7. Blunt plate geometries.

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This study is a review of finite difference methods for computing interacting boundary layers for subsonic and supersonic separated flows.

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